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ORIGINAL RESEARCH

Further results on the group inverse of some anti-triangular block matrices

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Abstract Suppose $\Re$ is a right Ore domain with unity 1. In this paper, we investigate the existence of the group inverse of some anti-triangular block matrices over $\Re$ and obtain the sufficient and necessary conditions for such existence. Further, the representations of the group inverse for the following two classes are given.

(i) $M = \left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right)$, where $CA = C$;

(ii) $M = \left(\begin{smallmatrix} A & B \\ B & 0 \end{smallmatrix}\right)$, where $B^\sharp$ exists and $BA = BAB^\sharp B$.

The results extend the earlier works of Liu et al. (in Appl. Math. Comput. 218:8978–8986, 2012) and Zhao et al. (in E. J. Linear Algebra 21:63–75, 2010). Some results in special cases are also generalized to any ring.

Keywords Right Ore domain · Block matrix · Group inverse · Ring

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1 Introduction

For a square matrix A, the matrix X is called the group inverse of A if X satisfies the matrix equations

$$AXA = A, XAX = X \quad \text{and} \quad AX = XA.$$

It is well known that if the group inverse X exists, it is unique, and is denoted by $A^{\sharp}$. Let A^{π} be $I - AA^{\sharp}$.

The group inverse of block matrix is a very useful tool in many fields, such as iterative methods, Markov chains, singular differential and difference equations, see [1–8].

In 1983, in the context of differential equations, Campbell et al. [9] proposed the problem of finding the representation for the Drazin inverse (group inverse) of the anti-triangular block matrix $\left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right)$. This problem remains open. However, there are many results in some special cases, see [10, 13–18, 20–22]. It is important to study them in a larger ring, see for example [11, 12, 19, 26].

Liu et al. [10] studied this problem under the conditions $A^2 = A, CA = C$ over complex fields. In this paper, we not only delete the condition $A^2 = A$ but also solve it for the matrices over right Ore domains by using matrix equations. This also generalizes the results of Ge et al. [27].

On the other hand, in [20], Zhao et al. characterized the existence and the representation of group inverse for block matrix over skew fields $\left(\begin{smallmatrix} A & B \\ B & 0 \end{smallmatrix}\right)$ under the condition that $B^{\sharp}$ exists and $BAB^{\pi} = 0$. In this paper, we extend these results to right Ore domains. Some results in special cases are studied over rings with unity 1.

In this paper, let $\mathbf{R}$ be a ring with unity 1. A ring is called a right Ore domain (denoted by $\Re$) if it possesses no zero divisors and every two elements of the ring have a right common multiple. Integral rings, polynomial rings in an indeterminate over field, noncommutative principal ideal domains and so on are right Ore domains. A left Ore domain is defined similarly. Every right (left) Ore domain $\Re$ can be embedded in the skew field (denoted by $K_{\Re}$) of quotients of itself. More details are found in [23–25]. Let $\Re^{m\times n}$ (respectively, $\mathbf{R}^{m\times n}$) be the set of all $m \times n$ matrices over $\Re$ (respectively, $\mathbf{R}$). The rank of a matrix $A \in \Re^{m\times n}$ (denoted by r(A)) is defined as the rank of A over $K_{\Re}$, i.e., the maximum order of all invertible subblocks of A over $K_{\Re}$. For convenience, we suppose the right Ore domain $\Re$ has identity 1.

2 Some lemmas

The following three lemmas will be used in the paper.

Lemma 1 [12] *Let $A \in \Re^{n\times n}$. Then the following are equivalent*:

(i) $A^{\sharp}$ *exists*;
(ii) $A^2X = A$ *for some* $X \in \Re^{n\times n}$. *In this case,* $A^{\sharp} = AX^2$;
(iii) $YA^2 = A$ *for some* $Y \in \Re^{n\times n}$. *In this case,* $A^{\sharp} = Y^2A$.

Lemma 2 [11, 26] *Let $A \in \mathbf{R}^{n\times n}$. Then the following are equivalent:*

(i) $A^\sharp$ *exists*;
(ii) $A^2X = A, YA^2 = A$ *for some* X, Y *over* $\mathbf{R}$. *In this case,* $A^\sharp = Y^2A = AX^2 = YAX$.

Lemma 3 *Let $A, B \in \mathbf{R}^{n\times n}$. If $BAB^\pi = 0$, $B^\sharp$ and $(AB^\pi)^\sharp$ exist, then*

(i) $B^\sharp AB^\pi = 0, (AB^\pi)^\sharp B = 0, B(AB^\pi)^\sharp = 0$;
(ii) $A(AB^\pi)^\sharp = (AB^\pi)^\sharp AB^\pi$;
(iii) $A(AB^\pi)^\sharp AB^\pi = AB^\pi$.

Proof (i) $B^\sharp AB^\pi = (B^\sharp)^2BAB^\pi = 0$, $(AB^\pi)^\sharp B = ((AB^\pi)^\sharp)^2AB^\pi B = 0$, similarly, $B(AB^\pi)^\sharp = 0$.

(ii) $A(AB^\pi)^\sharp = AB^\pi(AB^\pi)^\sharp = (AB^\pi)^\sharp AB^\pi$.

(iii) From (ii), we can easily show that (iii) holds. □

3 Main results

The following is the main result in this note.

Theorem 1 *Let $M = \begin{pmatrix} A & B \\ C & 0\end{pmatrix}$, where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{m\times n}$. If $CA = C$, then*

(i) $M^\sharp$ *exists if and only if* $(CB)^\sharp$ *and* $A^\sharp$ *exist,* $A^\pi B(CB)^\pi = 0$;
(ii) *If* $M^\sharp$ *exists, then* $M^\sharp = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4\end{pmatrix}$, *where*

$$M_1 = A^\sharp - (A^\sharp)^2B(CB)^\pi C - A^\sharp B(CB)^\sharp C - A^\sharp B(CB)^\pi C - A^\pi B[(CB)^\sharp]^2C,$$

$$M_2 = (A^\sharp)^2B(CB)^\pi + A^\pi B[(CB)^\sharp]^2 + A^\sharp B(CB)^\sharp + A^\pi B(CB)^\sharp,$$

$$M_3 = (CB)^\pi C + (CB)^\sharp C,$$

$$M_4 = -(CB)^\sharp.$$

Proof (i) The "only if" part.

Since $CA^2 = CA = C$, we have

$$\begin{pmatrix} I & B \\ 0 & I\end{pmatrix}\begin{pmatrix} I & 0 \\ C & I\end{pmatrix}\begin{pmatrix} A^2 & AB - BCB \\ 0 & CBCB\end{pmatrix} = \begin{pmatrix} A^2 + BC & AB \\ C & CB\end{pmatrix} = M^2 \tag{1}$$

By Lemma 1, $M^\sharp$ exists if and only if $M = YM^2$ for some $Y \in \Re^{(n+m)\times(n+m)}$. Let

$$Y\begin{pmatrix} I & B \\ 0 & I\end{pmatrix}\begin{pmatrix} I & 0 \\ C & I\end{pmatrix} = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4\end{pmatrix},$$

where $Y_1 \in \Re^{n\times n}$, so

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4\end{pmatrix}\begin{pmatrix} A^2 & AB - BCB \\ 0 & CBCB\end{pmatrix} = \begin{pmatrix} A & B \\ C & 0\end{pmatrix},$$

it follows that $Y_1A^2 = A$, by Lemma 1, so $A^\sharp$ exists.

According to Lemma 1, $M^\sharp$ exists if and only if there exists a matrix $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \in \Re^{(n+m)\times(n+m)}$ such that $M^2X = M$, where $X_1 \in \Re^{n\times n}$.

By (1), we have

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}\begin{pmatrix} A^2 & AB - BCB \\ 0 & CBCB \end{pmatrix} X = M.$$

Hence,

$$\begin{pmatrix} A^2 & AB - BCB \\ 0 & CBCB \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -C & I \end{pmatrix}\begin{pmatrix} I & -B \\ 0 & I \end{pmatrix} M.$$

Namely,

$$A^2X_1 + (AB - BCB)X_3 = A - BC, \tag{2}$$

$$A^2X_2 + (AB - BCB)X_4 = B, \tag{3}$$

$$CBCBX_3 = CBC, \tag{4}$$

$$CBCBX_4 = -CB. \tag{5}$$

From (5) and Lemma 1, we know $(CB)^\sharp$ exists. By (3), we have $-A^\pi BCBX_4 = A^\pi B$. i.e.,

$$-A^\pi B(CB)^\sharp CBCBX_4 = A^\pi B.$$

Substitute (5) into the above equation, we have $A^\pi B(CB)^\sharp CB = A^\pi B$. Therefore, $A^\pi B(CB)^\pi = 0$.

The "if" part.

Let $X_1 = A^\sharp - A^{\sharp^2} B(CB)^\pi C - A^\sharp B(CB)^\sharp C$, $X_2 = A^{\sharp^2} B(CB)^\pi + A^\sharp B(CB)^\sharp$, $X_3 = (CB)^\sharp C$, $X_4 = -(CB)^\sharp$.

Note that $A^\pi B(CB)^\pi = 0$. It is easy to verify that (2)–(5) hold. This implies $M = M^2X$ has a solution, so $M^\sharp$ exists.

(ii) By Lemma 1, $A^\pi B(CB)^\pi = 0$ and $CA^\sharp = CAA^\sharp = CA^2A^\sharp = CA = C$, the expression of $M^\sharp$ can be obtained from $M^\sharp = MX^2$. Next we can compute that

$$\begin{aligned}
M^\sharp &= \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \\
&= \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} A^\sharp - A^{\sharp^2} B(CB)^\pi C - A^\sharp B(CB)^\sharp C & A^{\sharp^2} B(CB)^\pi + A^\sharp B(CB)^\sharp \\ (CB)^\sharp C & -(CB)^\sharp \end{pmatrix} \\
&\quad \times \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \\
&= \begin{pmatrix} AA^\sharp - A^\sharp B(CB)^\pi C + A^\pi B(CB)^\sharp C & A^\sharp B(CB)^\pi - A^\pi B(CB)^\sharp \\ (CB)^\pi C & CB(CB)^\sharp \end{pmatrix}
\end{aligned}$$

$$\times \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$
$$= \begin{pmatrix} A^\sharp - A^{\sharp^2} B(CB)^\pi C - A^\sharp B(CB)^\sharp C & A^{\sharp^2} B(CB)^\pi + A^\pi B((CB)^\sharp)^2 \\ -A^\sharp B(CB)^\pi C - A^\pi B((CB)^\sharp)^2 C & +A^\sharp B(CB)^\sharp + A^\pi B(CB)^\sharp \\ (CB)^\pi C + (CB)^\sharp C & -(CB)^\sharp \end{pmatrix}$$
$$= \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}. \qquad \square$$

Example for Theorem 1: Let $\Re$ be the integer ring, and let $M = \left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right)$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 3 \\ 2 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

It is easy to verify $A^2 \neq A$, $CA = C$. Furthermore, $A^\sharp$ and $(CB)^\sharp$ exist. By computation,

$$A^\sharp = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A^\pi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (CB)^\sharp = CB = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix},$$
$$(CB)^\pi = \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix},$$

so $A^\pi B(CB)^\pi = 0$. By Theorem 1, $M^\sharp$ exists and

$$M^\sharp = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 7 & -1 & 0 & -13 & 3 \\ -2 & 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & -2 & 0 \end{pmatrix}.$$

Similarly, we can prove the counterpart of Theorem 1.

Theorem 2 *Let $M = \left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right)$, where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{m\times n}$. If $AB = B$, then*

(i) *$M^\sharp$ exists if and only if $(CB)^\sharp$ and $A^\sharp$ exist and $(CB)^\pi CA^\pi = 0$;*
(ii) *If $M^\sharp$ exists, then $M^\sharp = \left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & M_4 \end{smallmatrix}\right)$, where*

$$M_1 = A^\sharp - B(CB)^\pi CA^\sharp - B(CB)^\pi C(A^\sharp)^2 - B(CB)^\sharp CA^\sharp - B[(CB)^\sharp]^2 CA^\pi,$$
$$M_2 = B(CB)^\sharp + B(CB)^\pi,$$
$$M_3 = (CB)^\sharp CA^\sharp + (CB)^\sharp CA^\pi + [(CB)^\sharp]^2 CA^\pi + (CB)^\pi C(A^\sharp)^2,$$
$$M_4 = -(CB)^\sharp.$$

Next we consider a special case of Theorems 1 and 2, and investigate it over any ring.

Theorem 3 *Let* $M=\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, *where* $A\in \mathbf{R}^{n\times n}, B\in \mathbf{R}^{n\times m}, C\in \mathbf{R}^{m\times n}$. *If* $AB=B, CA=C$, *then*

(i) $M^\sharp$ *exists if and only if* $(CB)^\sharp$ *and* $A^\sharp$ *exist*;
(ii) *If* $M^\sharp$ *exists, then*

$$M^\sharp=\begin{pmatrix} A^\sharp-2B(CB)^\pi C-B(CB)^\sharp C & B(CB)^\sharp+B(CB)^\pi \\ (CB)^\sharp C+(CB)^\pi C & -(CB)^\sharp \end{pmatrix}.$$

Proof (i) The "only if" part.

If $M^\sharp$ exists, then by Lemma 2 there exist matrices X and Y over $\mathbf{R}$ such that $M=M^2X$ and $M=YM^2$.

By $AB=B$ and $CA=C$, we have

$$\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}\begin{pmatrix} A^2 & 0 \\ -CBC & CBCB \end{pmatrix}\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}=\begin{pmatrix} A^2+BC & B \\ C & CB \end{pmatrix}=M^2.$$

Let

$$X=\begin{pmatrix} I & 0 \\ -C & I \end{pmatrix}\begin{pmatrix} I & -B \\ 0 & I \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}.$$

Then

$$\begin{pmatrix} A^2 & 0 \\ -CBC & CBCB \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}=\begin{pmatrix} A & 0 \\ 0 & -CB \end{pmatrix}.$$

From above, we get

$$A^2X_1=A, \tag{6}$$

$$A^2X_2=0, \tag{7}$$

$$-CBCX_1+CBCBX_3=0, \tag{8}$$

$$-CBCX_2+CBCBX_4=-CB. \tag{9}$$

It is easy to get

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}\begin{pmatrix} A^2 & B-BCB \\ 0 & CBCB \end{pmatrix}=M^2.$$

From above equations and $M=YM^2$, let $Y=\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$, we have

$$Y_1A^2=A, \tag{10}$$

$$Y_1(B-BCB)+Y_2CBCB=B, \tag{11}$$

$$Y_3A^2=C, \tag{12}$$

$$Y_3(B-BCB)+Y_4CBCB=0. \tag{13}$$

By Lemma 2, (6) and (10) imply $A^\sharp$ exists.

By (7) and (12), we have $CX_2 = 0$ and $Y_3B = CB$. Substitute these identities into (9) and (13) respectively, we get $CBCBX_4 = -CB$ and $CB = (I - Y_4)CBCB$.

By Lemma 1, $(CB)^\sharp$ exists.

The 'if' part. Let

$$X_1 = A^\sharp, X_2 = 0, X_3 = (CB)^\sharp C, X_4 = -(CB)^\sharp.$$

$$Y_1 = A^\sharp, Y_2 = B(CB)^\sharp, Y_3 = C, Y_4 = CB(CB)^\sharp - (CB)^\sharp.$$

It is easy to verify (6)–(13) hold. That implies $M = M^2X$ and $M = YM^2$ have solutions.

From Lemma 2, we know $M^\sharp$ exists.

(ii) By Lemma 2, the expression of $M^\sharp$ can get from $M^\sharp = YMX$. $\square$

Example for Theorem 3: Let Z be the integer ring, and let $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ be a matrix over $Z/(6Z)$, where

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 1 \\ 3 & 3 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix}.$$

It is easy to verify that $A^2 \neq A$, $AB = B$, $CA = C$. Furthermore, $A^\sharp$ and $(CB)^\sharp$ exist.

By computation,

$$A^\sharp = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (CB)^\sharp = CB = \begin{pmatrix} 0 & 0 \\ 3 & 3 \end{pmatrix}.$$

By Theorem 3, $M^\sharp$ exists and

$$M^\sharp = \begin{pmatrix} 1 & 1 & 0 & 3 & 1 \\ 0 & -2 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 3 & 3 \end{pmatrix}.$$

Remark 1 From Theorem 1 and 2, we can obtain Theorem 3.1 and 3.2 of [27] and Theorem 2.1 and 2.2 of [10]. From Theorem 3, we can also obtain the Corollary 2.2 of [10]. In above two examples, we especially point that $A^2 \neq A$. This shows that the generalizations are true.

The following results extend the corresponding works of Zhao et al. [20].

Theorem 4 *Let* $M = \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}$, *where* $A, B \in \Re^{n\times n}$. *If* $B^\sharp$ *exists and* $BAB^\pi = 0$. *Then*

(i) $M^\sharp$ *exists if and only if* $(AB^\pi)^\sharp$ *exists.*

(ii) *If* $M^\sharp$ *exists, then* $M^\sharp = \left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & M_4 \end{smallmatrix}\right)$, *where*

$$
\begin{aligned}
M_1 &= B^\pi A(B^\sharp)^2 - (AB^\pi)^\sharp AB^\pi A(B^\sharp)^2 + (AB^\pi)^\sharp;\\
M_2 &= -B^\pi A(B^\sharp)^2 AB^\sharp + (AB^\pi)^\sharp AB^\pi A(B^\sharp)^2 AB^\sharp - (AB^\pi)^\sharp AB^\sharp + B^\sharp;\\
M_3 &= B^\sharp;\\
M_4 &= -B^\sharp AB^\sharp.
\end{aligned}
$$

Proof The "only if" part of (i).

It is easy to get

$$
M = \begin{pmatrix} A & B \\ B & 0 \end{pmatrix} = \begin{pmatrix} B^\pi A & B \\ B & 0 \end{pmatrix} \begin{pmatrix} I & O \\ B^\sharp A & I \end{pmatrix},
$$

$$
M^2 = \begin{pmatrix} A^2 + B^2 & AB \\ BA & B^2 \end{pmatrix} = \begin{pmatrix} AB^\pi A + B^2 & AB \\ 0 & B^2 \end{pmatrix} \begin{pmatrix} I & O \\ B^\sharp A & I \end{pmatrix}.
$$

Since $M^\sharp$ exists, from Lemma 1 we know $YM^2 = M$ has a solution.
Let $Y = \left(\begin{smallmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{smallmatrix}\right)$.
We have

$$
Y_1 AB^\pi A + Y_1 B^2 = B^\pi A, \tag{14}
$$

$$
Y_1 AB + Y_2 B^2 = B, \tag{15}
$$

$$
Y_3 AB^\pi A + Y_3 B^2 = B, \tag{16}
$$

$$
Y_3 AB + Y_4 B^2 = 0. \tag{17}
$$

By (14), we have $Y_1(AB^\pi)^2 = B^\pi AB^\pi = AB^\pi$. From Lemma 1, we know $(AB^\pi)^\sharp$ exists. Next, we prove the sufficiency of (i) and the expression of (ii):

Let $X = \left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & M_4 \end{smallmatrix}\right)$. By Lemma 3, the sufficiency of (i) and the expression of $M^\sharp$ are similar to the proof in [20].

$$
MX = \begin{pmatrix} A(AB^\pi)^\sharp + BB^\sharp & -A(AB^\pi)^\sharp AB^\sharp + B^\pi AB^\sharp \\ 0 & BB^\sharp \end{pmatrix} = XM.
$$

It is easy to verify that $XMX = M$, $MXM = M$.
So $X = M^\sharp$. □

Similarly, we state the symmetrical result of Theorem 4.

Theorem 5 *Let* $M = \left(\begin{smallmatrix} A & B \\ B & 0 \end{smallmatrix}\right)$, *where* $A, B \in \Re^{n\times n}$. *If* $B^\sharp$ *exists and* $B^\pi AB = 0$, *then*

(i) $M^\sharp$ *exists if only if* $(B^\pi A)^\sharp$ *exists.*

(ii) *If $M^\sharp$ exists, then* $M^\sharp = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, *where*

$$M_1 = (B^\sharp)^2 AB^\pi - (B^\sharp)^2 AB^\pi A(B^\pi A)^\sharp + (B^\pi A)^\sharp;$$
$$M_2 = B^\sharp;$$
$$M_3 = -B^\sharp A(B^\sharp)^2 AB^\pi + B^\sharp A(B^\sharp)^2 AB^\pi A(B^\pi A)^\sharp - B^\sharp A(B^\pi A)^\sharp + B^\sharp;$$
$$M_4 = -B^\sharp AB^\sharp.$$

Proof The proof is similar to Theorem 4, so we omit it. □

Next we consider a special case of Theorem 4 and 5, and investigate it over any ring.

Theorem 6 *Let* $M = \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}$, *where* $A, B \in \mathbf{R}^{n\times n}$. *If* $B^\sharp$ *exists and* $BAB^\pi = 0$, $B^\pi AB = 0$, *then*

(i) $M^\sharp$ *exists if and only if* $(AB^\pi)^\sharp$ *exists.*
(ii) *If* $M^\sharp$ *exists, then*

$$M^\sharp = \begin{pmatrix} (AB^\pi)^\sharp & B^\sharp \\ B^\sharp & -B^\sharp AB^\sharp \end{pmatrix}.$$

Proof The "only if" part of (i).

Let
$$M^2 = \begin{pmatrix} I & AB^\sharp \\ 0 & I \end{pmatrix} \begin{pmatrix} AB^\pi A + B^2 & 0 \\ 0 & B^2 \end{pmatrix} \begin{pmatrix} I & O \\ B^\sharp A & I \end{pmatrix}.$$

The decomposition of M is the same as in Theorem 4. By Lemma 2, $M^\sharp$ exists if and only if there exist X, Y over $\mathbf{R}$ such that $MX^2 = M$ and $YM^2 = M$.

Let
$$X = \begin{pmatrix} I & 0 \\ -B^\sharp A & I \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, \qquad Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \begin{pmatrix} I & -AB^\sharp \\ 0 & I \end{pmatrix},$$

then we have

$$AB^\pi = \left(AB^\pi A + B^2\right)X_1, \tag{18}$$

$$B = \left(AB^\pi A + B^2\right)X_2, \tag{19}$$

$$B = B^2 X_3, \tag{20}$$

$$0 = B^2 X_4, \tag{21}$$

$$Y_1 AB^\pi A + Y_1 B^2 = B^\pi A, \tag{22}$$

$$Y_2 B^2 = B, \tag{23}$$

$$Y_3 AB^\pi A + Y_3 B^2 = B, \tag{24}$$

$$Y_4 B^2 = 0. \tag{25}$$

By (18) and (22), we have $AB^{\pi}AB^{\pi}X_1 = B^{\pi}AB^{\pi}AB^{\pi}X_1 = B^{\pi}AB^{\pi}AX_1 = B^{\pi}AB^{\pi} = AB^{\pi}$ and $Y_1 AB^{\pi}AB^{\pi} = AB^{\pi}$, respectively. From Lemma 2, we know $(AB^{\pi})^{\sharp}$ exists. Next, we prove the sufficiency of (i) and the expression of (ii).

Let

$$X = \begin{pmatrix} (AB^{\pi})^{\sharp} & B^{\sharp} \\ B^{\sharp} & -B^{\sharp}AB^{\sharp} \end{pmatrix}.$$

By Lemma 3,

$$MX = \begin{pmatrix} A(AB^{\pi})^{\sharp} + BB^{\sharp} & 0 \\ 0 & BB^{\sharp} \end{pmatrix} = XM.$$

It is easy to verify that $XMX = M$, $MXM = M$.

So $X = M^{\sharp}$. □

Remark 2 We have expressed Theorems 1–2 and 4–5 over right Ore domains, but how to solve them in rings? This is still an open question.

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